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LETTER TO THE EDITOR

Critical exponents of directed self-avoiding walks

John L Cardy

Department of Physics, University of California, Santa Barbara, California 93106, USA

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Abstract. It is argued that directed self-avoiding random walks are in the same universality class as directed unrestricted random walks, and therefore should exhibit mean-field exponents $\gamma = 1$, $\nu_{\parallel} = 1$, $\nu_{\perp} = \frac{1}{2}$ in all dimensionalities.

In a recent letter, Chakrabarti and Manna (1983) have introduced the problem of directed self-avoiding random walks (SAW), and discussed some exact and numerical results in two dimensions. In this note we formulate the field-theoretic description of this problem, in analogy with the work of de Gennes (1972) on the ordinary SAW, and show that it has simplifying features which imply that the critical exponents are mean-field like for all dimensionalities.

In the directed problem, there is a preferred direction (labelled by a coordinate $r_{\parallel}$), and walkers are allowed to proceed only in the direction of increasing $r_{\parallel}$. (In fact, the problem where walkers can also move in the opposite direction, but steps with $\delta r_{\parallel} > 0$ and $\delta r_{\parallel} < 0$ have unequal weight, will be in the same universality class.) Let $G_N(\boldsymbol{r})$ be the number of such SAW from the origin to $\boldsymbol{r}$ with N steps, and introduce the generating function

$$G(\boldsymbol{r}, x) = \sum_{N=0}^{\infty} G_N(\boldsymbol{r}) x^N. \tag{1}$$

This may be written as an integral over an n-component *complex* field $\varphi_\alpha(\boldsymbol{r})$:

$$G(\boldsymbol{r}, x) = \lim_{n\to 0} \int \prod_{\boldsymbol{r},\alpha} \mathrm{d}\varphi_\alpha(\boldsymbol{r}) \varphi_1(\boldsymbol{r}) \varphi_1^*(\boldsymbol{0}) \mathrm{e}^{-S} \tag{2}$$

where

$$S = \sum_{\boldsymbol{r},\boldsymbol{r}',\alpha} \varphi_\alpha^*(\boldsymbol{r}) G_0^{-1}(\boldsymbol{r}-\boldsymbol{r}') \varphi_\alpha(\boldsymbol{r}') + \frac{u_0}{4} \sum_{\boldsymbol{r}} \left(\sum_\alpha \varphi_\alpha^*(\boldsymbol{r}) \varphi_\alpha(\boldsymbol{r}) \right)^2. \tag{3}$$

The quartic term provides a repulsive potential, and the limit $n \to 0$ eliminates closed loops, as usual. The Fourier transform of the bare propagator G_0 has the form, in the long-wavelength limit,

$$G_0^{-1}(\boldsymbol{k}) = c_1 - \mathrm{i} c_2 k_{\parallel} + c_3 k_{\parallel}^2 + c_4 \boldsymbol{k}_{\perp}^2 + \ldots \tag{4}$$

where the c_i are constants dependent on x and the lattice structure. For ordinary SAW, $c_2 = 0$, and we regain the result of de Gennes (1972), since the n-component complex field is equivalent to a $2n$-component real field.

The massless limit of the field theory (3), when the renormalised value of c_1 vanishes, occurs when $x = x_c$, the value at which (1) diverges. When $c_2 \neq 0$, the perturbation expansion may be expressed in terms of advanced and retarded propagators: $G_0 = G_0^{\rm adv} + G_0^{\rm ret}$, where

$$G_0^{\rm ret} \simeq (-\mathrm{i}c_2 k_\parallel + c_1 + c_4 \boldsymbol{k}_\perp^2)^{-1} \tag{5}$$

$$G_0^{\rm adv} \simeq (\mathrm{i}c_2 k_\parallel + c_2^2 c_3^{-1} + c_1 + c_4 \boldsymbol{k}_\perp^2)^{-1} \tag{6}$$

in the limit where c_1 is small. We see that only the retarded propagator becomes massless when $c_1 \to 0$. But the loop corrections to the full correlation function (2) must involve at least the advanced propagator in each loop, and so they are always infrared *finite* even as $c_1 \to 0$. Hence the interaction term u_0 is irrelevant and we should always observe mean-field exponents. Physically, this is because the overwhelming majority of directed walks will never approach self-intersection.

According to (5) then, asymptotically,

$$G(\boldsymbol{r}, x) \sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{-\mathrm{i}c_2 k_\parallel + c_1 + c_4 \boldsymbol{k}_\perp^2} \tag{7}$$

where the renormalised values of c_1, c_2, c_4 are implied, so that $c_1 \propto (x - x_c)$. From (7) the mean-field exponents may be obtained. We see that $G(\boldsymbol{k}=0) \propto (x-x_c)^{-1}$, so $\gamma = 1$, in agreement with Chakrabarti and Manna (1983). However, in discussing the average end-to-end distance of the walks one must distinguish two length scales, as in directed percolation (Cardy and Sugar 1980) and in directed animals (Redner and Yang 1982). The end-to-end distance parallel and perpendicular to the preferred direction should scale as $N^{\nu_\parallel}$ and $N^{\nu_\perp}$ respectively, where, from (7), $\nu_\parallel = 1$, $\nu_\perp = \frac{1}{2}$. Chakrabarti and Manna (1983) do not make such a distinction, and obtain an intermediate value, which, in the context of our theory, must only be an effective exponent.

Finally, we note that while the asymptotic behaviour of $G_N(\boldsymbol{r})$ is trivial in this model, other quantities are more interesting. For example, the number of distinct *pairs* of SAW from $\mathbf{0}$ to $\boldsymbol{r}$, with a total of N steps, $G_N^{(2)}(\boldsymbol{r})$, is related to the correlation function $\langle \varphi_1(\boldsymbol{r})\varphi_2(\boldsymbol{r})\varphi_1^*(\mathbf{0})\varphi_2^*(\mathbf{0})\rangle$, which in the limit in which only the retarded piece of the propagator is kept, is represented by the sum of diagrams shown in figure 1.

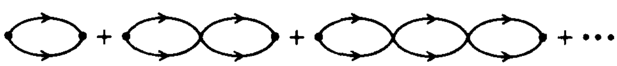

Figure 1. Diagrams contributing to $G^{(2)}(\boldsymbol{r}, x)$. The replica index structure is not indicated.

We then find

$$\sum_{\boldsymbol{r}} G^{(2)}(\boldsymbol{r}, x) \sim \begin{matrix} (x-x_c)^{D/2-1} & (D>2) \\ u_0^{-2}(x-x_c)^{1-D/2} & (D<2) \end{matrix} \tag{8}$$

where D is the number of transverse dimensions. This yields the total number of such pairs, $\Sigma_{\boldsymbol{r}} G_N^{(2)}(\boldsymbol{r}) \sim x_c^{-N} N^{\gamma_2 - 1}$ where $\gamma_2 = -|D/2-1|$ for $D \neq 2$. When $D=2$, the asymptotic behaviour is $x_c^{-N}/N(\ln N)^2$.

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